The maximal quantity in the proposed theory is $\sigma_u = k_1 + k_2$, and is generally dependent on the cosine of the angle between the vectors e_{ij}' and de_{ij}'. The relationships of classical ideal plasticity theory occur from (3.11) in the particular case with $k_1 = 0$, $k_2 \neq 0$, and of deformation ideal plasticity theory for $k_1 \neq 0$, $k_2 = 0$.

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DYNAMICS OF FREE SYSTEMS OF MATERIAL POINTS WITH ELASTIC CONSTRAINTS

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S. A. ALEKSEEV
(Moscow)
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A method is indicated for constructing the equations of motion of free systems of material points connected to each other by inertialess elastic constraints. The system configuration is arbitrary, and any arbitrarily time-dependent external forces are applied to the material points.

1. Rigid system. Let us assume that there are N material points in the system. Let m_i denote the mass of the ith point, M the sum of all the masses so that

$$M = \Sigma m_i \tag{1.1}$$

Here, as everywhere below, the symbol Σ denotes summation over all material points of the system, i.e. over i between 1 and N.

We call a system for which the deformations of the elastic constraints are zero – a rigid system. If the constraints are absolutely rigid, the rigid system is substantially an absolutely rigid body.

Let us refer the rigid system to fixed Cartesian coordinates with the unit vectors $\mathbf{e}_j^\circ$ $(j = 1,2,3)$, and also to moving coordinates with $\mathbf{e}_j$ $(j = 1,2,3)$ directed along the principal central axes of inertia of the rigid system. If ρ_i°, ρ_c° are radius-vectors of the ith point and the center of mass of the rigid system relative to the origin of the fixed coordinates, and

$$\rho_i = x_i^1\mathbf{e}_1 + x_i^2\mathbf{e}_2 + x_i^3\mathbf{e}_3 \tag{1.2}$$

is the radius-vector of the ith point of a rigid system relative to its center of mass, then

$$\rho_i^\circ = \rho_c^\circ + \rho_i \tag{1.3}$$

By the definition of the center of mass

$$\rho_c{}^\circ = \frac{1}{M} \Sigma m_i \rho_i{}^\circ, \qquad \Sigma m_i \rho_i = 0 \tag{1.4}$$

2. Elastic displacements. In contrast to an attached system, the elastic displacements in a free system can be introduced differently. For example, the elastic displacement U_i can be defined as the deviation of the ith point from its position in a rigid system which is fixed or moving according to the given law

$$\mathbf{r}_i{}^\circ = \rho_c{}^\circ + \rho_i + \mathbf{U}_i \tag{2.1}$$

Here $\mathbf{r}_i{}^\circ$ is the radius-vector of the ith point of an elastic system.

However, it is more expedient to define the elastic displacement $\mathbf{V}_i$ of the ith point as its deviation from the position in the rigid system whose position in space is unknown in advance, and to impose two homogeneous vector conditions simultaneously on $\mathbf{V}_i$

$$\Sigma m_i \mathbf{V}_i = 0, \qquad \Sigma m_i \mathbf{r}_i \times \mathbf{V}_i = 0 \tag{2.2}$$

Here

$$\mathbf{r}_i{}^\circ = \mathbf{r}_c{}^\circ + \mathbf{r}_i + \mathbf{V}_i, \qquad \mathbf{r}_i = x_i{}^1\mathbf{e}_1' + x_i{}^2\mathbf{e}_2' + x_i{}^3\mathbf{e}_3' \tag{2.3}$$

where $\mathbf{r}_i{}^\circ$ is the radius-vector of the center of mass of the elastic system, $\mathbf{r}_i$ the radius-vector of the ith point of the rigid system relative to its center of mass, and the basis $\mathbf{e}_j'$ is not known in advance. The coordinates $x_i{}^k$ have the same values as in (1.2).

The generality of (2.1) and (2.3) is identical. This is easily shown for small elastic displacements, and under the assumption that the basis $\mathbf{e}_j'$ is obtained from the basis $\mathbf{e}_j$ by rotation through a small angle θ. The last assumption does not diminish the generality of the deduction since it can be considered that the assigned motion of the basis $\mathbf{e}_j$ is close to the motion of the basis $\mathbf{e}_j'$; the smallness of the elastic displacements is essential and will be utilized later.

Setting $\mathbf{r}_i = \rho_i + \theta \times \rho_i$, we equate the right sides of (2.1) and (2.3.1); we hence obtain

$$\mathbf{V}_i = -\mathbf{r}_c{}^\circ + \rho_c{}^\circ - \mathbf{r}_i + \rho_i + \mathbf{U}_i$$

It follows from (1.4.2),(1.2) and (2.3.2) that

$$\Sigma m_i \mathbf{r}_i = 0 \tag{2.4}$$

Utilizing this, $\mathbf{r}_c{}^\circ$ and θ can be expressed explicitly in terms of $\rho_c{}^\circ$, ρ_i, $\mathbf{U}_i$. No constraints are hence imposed on either $\mathbf{U}_i$ or $\mathbf{V}_i$. The equivalent generality of both representations of the elastic displacements is hereby proved.

If all the constraints in the system are elastic, i.e. if there are no absolutely rigid constraints among them, the system has $3N$ degrees of freedom. Six of them refer to the motion of the basis $\mathbf{e}_j'$ and the remaining $2N-6$ to the elastic motion (for a linear system all of whose points are located on one line, five degrees of freedom correspond to the motion of the basis $\mathbf{e}_j'$ and $3N-5$ to the elastic motion).

Let us introduce generalized elastic coordinates $q_\lambda(t)$ as follows:

$$\mathbf{V}_i = q_\lambda \mathbf{b}_{i\lambda} \tag{2.5}$$

Here, as everywhere below, summation is assumed over repeated Greek subscripts, i.e. over all the degrees of freedom in the elastic motion. The vectors $\mathbf{b}_{i\lambda}$ (see Sect. 3) are constructed so that they satisfy the equalities

$$\Sigma m_i \mathbf{b}_{i\lambda} = 0, \qquad \Sigma m_i \mathbf{r}_i \times \mathbf{b}_{i\lambda} = 0 \tag{2.6}$$

Setting $q_\lambda = 1$, $q_\mu = 0$ $(\mu \neq \lambda)$ into (2.5), we obtain $\mathbf{V}_i = \mathbf{b}_{i\lambda}$. Hence, the vectors $\mathbf{b}_{i\lambda}$ can be called the unit elastic displacements.

3. Unit elastic displacements. Let us prescribe as many linearly independent groups of forces $\mathbf{f}_{i\lambda}$ as the system has degrees of freedom in elastic motion, where the forces $\mathbf{f}_{i\lambda}$ are statically equivalent to zero

$$\Sigma \mathbf{f}_{i\lambda} = 0, \qquad \Sigma \mathbf{r}_i \times \mathbf{f}_{i\lambda} = 0 \tag{3.1}$$

Linear independence means that there are no numbers g_λ, at least one of which is not zero, such that $g_\lambda \mathbf{f}_{i\lambda} = 0$.

Let us show how the $\mathbf{f}_{i\lambda}$ can be prescribed for some systems, where this process is simplest. A formalized method of evaluating the $\mathbf{f}_{i\lambda}$ for systems of arbitrary form is indicated in Sect. 7.

We direct the unit vector $\mathbf{e}_1'$ in a linear system along a line on which all the material points lie, and the unit vectors $\mathbf{e}_2'$, $\mathbf{e}_3'$ in some orthogonal directions. Applying forces of equal modulus but opposite direction to each pair of adjacent points, we obtain $N - 1$ groups of forces $\mathbf{f}_{i\lambda}$. Furthermore, we apply three forces statically equivalent to zero parallel to $\mathbf{e}_2'$ to every three adjacent points, and we obtain $N - 2$ groups of forces $\mathbf{f}_{i\lambda}$. The same operation in the plane containing $\mathbf{e}_3'$ and $\mathbf{e}_1'$ results in the construction of $N - 2$ groups of forces $\mathbf{f}_{i\lambda}$. In all we obtain $3N - 5$ groups of forces $\mathbf{f}_{i\lambda}$, which is how many degrees of freedom the system has in elastic motion.

In a plane system all of whose points are located in one plane and no three points lie on one straight line, we apply forces of equal modulus and opposite direction to each pair of points whose numbers differ by one and two units, and we obtain $2N - 3$ groups of forces $\mathbf{f}_{i\lambda}$ lying in the plane of the system. In order to obtain $N - 3$ groups of forces $\mathbf{f}_{i\lambda}$ of the normal planes of the system, let us consider four arbitrary points and let us apply a group of forces, statically equivalent to zero, to them. Then we form a new quartet of points, one of which is not in the first quartet, and we apply a system of forces equivalent to zero to these four. Extracting further quartets of points so that one point in each is in none of the previous quartets, we obtain $N - 3$ groups of forces $\mathbf{f}_{i\lambda}$.

In a three-dimensional system in which no three points lie on one straight line, we apply equal and opposite forces to pairs of points whose numbers differ by one, two and three units, and we obtain $3N - 6$ groups of forces $\mathbf{f}_{i\lambda}$ which possess the required properties.

Let us form the vectors

$$\boldsymbol{\beta}_{i\lambda} = \frac{M}{m_i} \mathbf{f}_{i\lambda} \tag{3.2}$$

Orthonormalizing the $\boldsymbol{\beta}_{i\lambda}$ we obtain the unitary displacements $\mathbf{b}_{i\lambda}$. Let us require that

$$\Sigma m_i \mathbf{b}_{i\lambda} \mathbf{b}_{i\mu} = \begin{cases} M, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases} \tag{3.3}$$

Let us apply the Gram-Schmidt process to orthonormalize the $\boldsymbol{\beta}_{i\lambda}$ with weights m_i and let us use the representation

$$\boldsymbol{\beta}_{i\lambda} = C_{1\lambda} \mathbf{b}_{i1} + C_{2\lambda} \mathbf{b}_{i2} + \ldots + C_{\lambda\lambda} \mathbf{b}_{i\lambda} \tag{3.4}$$

The scalars $C_{\mu\lambda}$ are determined from the conditions (3. 3).

The system of unitary displacements obtained is complete by construction so that any vectors $\mathbf{B}_i$ satisfying the conditions

$$\Sigma m_i \mathbf{B}_i = 0, \qquad \Sigma m_i \mathbf{r}_i \times \mathbf{B}_i = 0$$

can be represented as linear combinations of $\mathbf{b}_{i\lambda}$.

Let us note that the unit displacements depend on the configuration of the rigid system, and on the mass distributed therein, but are independent of the stiffnesses of the elastic constraints.

In determining the $\mathbf{b}_{i\lambda}$ by the method described here there is no need to determine the position of the center of mass and the directions of the principal axes of inertia; in the general case these operations are necessary.

4. Kinetic energy. We obtain the velocity of the ith point by differentiating the radius vector $\mathbf{r}^0_i$ with respect to the time t in conformity with (2.3.1)

$$\frac{d\mathbf{r}_i^\circ}{dt}=\mathbf{r}_c^{\circ\cdot}+\frac{d\mathbf{r}_i}{dt}+\frac{d\mathbf{V}_i}{dt} \tag{4.1}$$

The first member on the right side is the velocity of the center of mass of the elastic system. Moreover

$$\frac{d\mathbf{r}_i}{dt}=\mathbf{r}_i^{\cdot}+\boldsymbol{\omega}\times\mathbf{r}_i$$

Here the first member in the right side is not zero if the configuration of the rigid system changes with time. Let us consider the configuration of the rigid system to be invariant, and therefore, let us set $\mathbf{r}^{\cdot}_i = 0$.

Let us consider the angular velocity of the rigid system ω to be small. Let us neglect the small displacement $\mathbf{V}_i$ in the expression for the velocity of the ith point

$$\frac{d\mathbf{r}_i^\circ}{dt}=\mathbf{r}_c^{\circ\cdot}+\boldsymbol{\omega}\times(\mathbf{r}_i+\mathbf{V}_i)+\mathbf{V}_i^{\cdot}$$

as compared with the finite quantity $\mathbf{r}_i$ and let us arrive at the following expression:

$$\frac{d\mathbf{r}_i^\circ}{dt}=\mathbf{r}_c^{\circ\cdot}+\boldsymbol{\omega}\times\mathbf{r}_i+\mathbf{V}_i^{\cdot} \tag{4.2}$$

It is easy to obtain the equalities

$$\Sigma m_i\mathbf{V}_i^{\cdot}=0,\qquad \Sigma m_i\mathbf{r}_i\times\mathbf{V}_i^{\cdot}=0 \tag{4.3}$$

from (2.2), which denote that under the definition of the elastic displacements taken here the momentum and kinetic moment relative to the center of mass of the elastic motion are zero.

Taking account of (4.3) as well as (2.4), let us represent the kinetic energy of the elastic system as $T = {}^1/_2M\mathbf{r}_c^{\circ\cdot 2} + {}^1/_2\Sigma m_i(\boldsymbol{\omega}\times\mathbf{r}_i)^2 + {}^1/_2\Sigma m_i\mathbf{V}_i^{\cdot 2}$

By using (2.5) and (3.3) we transform this as follows:

$$T = {}^1/_2M\mathbf{r}_c^{\circ\cdot 2} + {}^1/_2\Sigma m_i(\boldsymbol{\omega}\times\mathbf{r}_i)^2 + {}^1/_2Mq_\lambda^{\cdot 2} \tag{4.4}$$

The first two members in the right side are the kinetic energy of the rigid system, the last member is the kinetic energy of elastic motion.

The generalized momenta in the elastic motion are

$$\frac{\partial T}{\partial q_\lambda^{\cdot}}=Mq_\lambda^{\cdot} \tag{4.5}$$

The generalized momenta in the motion of the basis $\mathbf{e}_j'$ are calculated exactly in the same manner as in the motion of a solid.

5. Potential energy and equations of motion. Let us define the unit forces

$$\mathbf{p}_{i\lambda}=\frac{m_i}{M}\,\mathbf{b}_{i\lambda} \tag{5.1}$$

It is easy to see that the $\mathbf{p}_{i\lambda}$ are statically equivalent to zero.

Let us note that the forces $\mathbf{p}_{i\lambda}$ produce work on the unit displacements $\mathbf{b}_{i\lambda}$ but not on the $\mathbf{b}_{i\mu}(\mu \neq \lambda)$. This follows from (3.3).

Let us apply some system of forces $\mathbf{P}_i$, statically equivalent to zero, which must be a linear combination of the forces $\mathbf{p}_{i\lambda}$

$$\mathbf{P}_i = s_\lambda \mathbf{p}_{i\lambda} \tag{5.2}$$

to the points of the system.

The application of the forces $\mathbf{P}_i$ causes the appearance of elastic displacements $\mathbf{V}_i = q_\lambda \mathbf{b}_{i\lambda}$. The work of the force $\mathbf{P}_i$ on the displacements $\mathbf{V}_i$ equals the strain potential energy of the system

$$\Pi = {}^1/_2 \Sigma s_\lambda q_\mu \mathbf{p}_{i\lambda} \mathbf{b}_{i\mu}, \qquad \Pi = {}^1/_2 s_\lambda q_\lambda \tag{5.3}$$

Here (5.2), (2.5) and (3.3) are taken into account.

Applying the forces $\mathbf{p}_{i\lambda}$ to the elastic system we calculate the displacements

$$q_\lambda = \delta_{\lambda\mu} s_\mu \tag{5.4}$$

Here $\delta_{\lambda\mu}$ are the Maxwell coefficients [1, 2]. If the elastic constraints are rods, the $\delta_{\lambda\mu}$ are evaluated by the Mohr formula [1, 2], which can be written symbolically as follows:

$$\delta_{\lambda\mu} = \int \frac{T_\lambda T_\mu}{B} ds$$

Here T_λ, T_μ are the states of stress in the elastic constraints upon the application of the unit forces $\mathbf{p}_{i\lambda}$ and $\mathbf{p}_{i\mu}$, B, respectively, of the constraint stiffnesses. The Mohr formula can be utilized even when the elastic constraints contain not only rods but also plates. The Maxwell coefficients are evaluated by applying a static loading.

Solving (5.4) for s_μ, we obtain $s_\mu = c_{\mu\lambda} q_\lambda$, where $c_{\mu\lambda}$ are the influence numbers which can be obtained as elements of the matrix $c = \| c_{\mu\lambda} \|$ inverse to the Maxwell matrix $\delta = \| \delta_{\lambda\mu} \|$. Substituting the values found for s into (5.4), we obtain

$$\Pi = {}^1/_2\, c_{\lambda\mu} q_\lambda q_\mu \tag{5.5}$$

If the potential energy of the system depends on its position in some force field, components corresponding to the potential energy of a solid, and therefore, independent of the elastic generalized coordinates q_λ, are introduced into the expression for Π.

The generalized forces in elastic motion are

$$Q_\lambda = \Sigma \mathbf{R}_i \frac{\partial \mathbf{r}_i^\circ}{\partial q_\lambda}, \qquad Q_\lambda = \Sigma \mathbf{R}_i \mathbf{b}_{i\lambda} \tag{5.6}$$

Here (2.3.1) and (2.6) have been taken into account, and $\mathbf{R}_i$ are external forces. The generalized forces corresponding to motion of the basis $\mathbf{e}'_j$ are determined exactly in the same manner as in motion of a solid.

According to (4.5) and (5.5) the equations of elastic motion are

$$M q_\lambda^{\cdot\cdot} + c_{\lambda\mu} q_\mu = Q_\lambda \tag{5.7}$$

The equations of motion of the basis $\mathbf{e}_j'$ do not differ from the equations of motion of a solid.

If the external forces are independent of the elastic displacement, the equations of motion of the basis $\mathbf{e}_j'$, and the equations of elastic motion separate. If it is necessary to take account of energy dissipation, a dissipation function can be introduced [3].

The constraint must be considered absolutely rigid if there are no elastic displacements upon applying the forces $\mathbf{p}_{i\lambda}$ according to (5.1). If the system has k rigid constraints, then its total number of degrees of freedom is $3N - k$, there are just as many equations of motion. There is evidently no need to construct forces $\mathbf{p}_{i\lambda}$ corresponding to rigid constraints, and to determine the Maxwell coefficients from them.

6. Independence of the elastic motion from the selection of the unit elastic displacements. The vectors $\mathbf{b}_{i\lambda}$ can be chosen differently for the same system. Let us show that the elastic motion is independent of the selection of $\mathbf{b}_{i\lambda}$.

Besides the complete system $\mathbf{b}_{i\lambda}$ let there be another complete system $\mathbf{b}_{i\lambda}^*$ which can be represented as a linear combination of the $\mathbf{b}_{i\lambda}$

$$\mathbf{b}_{i\lambda}^* = a_{\lambda\mu}\mathbf{b}_{i\mu} \tag{6.1}$$

Both systems of vectors $\mathbf{b}_{i\lambda}$ and $\mathbf{b}_{i\lambda}^*$ are orthonormalized according to (3. 3). From the equalities

$$\Sigma m_i \mathbf{b}_{i\lambda}^* \mathbf{b}_{i\mu}^* = a_{\lambda\nu} a_{\mu\rho} \Sigma m_i \mathbf{b}_{i\nu} \mathbf{b}_{i\rho}$$

we have, by taking account of (3. 3)

$$a_{\lambda\nu} a_{\mu\nu} = \begin{cases} 1, & \lambda = \mu \\ 0, & \lambda \neq \mu \end{cases} \tag{6.2}$$

Let us introduce the matrix $a = \| a_{\lambda\mu} \|$ and let us represent (6. 2) as $aa' = E$, where a' is the transpose of a, and E is the unit matrix. Hence $a' = a^{-1}$, so that the matrix a is orthogonal. The forces $\mathbf{p}_{i\lambda}$ and $\mathbf{p}_{i\lambda}^*$ generated by the vectors $\mathbf{b}_{i\lambda}$ and $\mathbf{b}_{i\lambda}^*$ are connected by the equalities

$$\mathbf{p}_{i\lambda}^* = a_{\lambda\mu}\mathbf{p}_{i\mu}$$

There is a dependence

$$\delta_{\lambda\mu}^* = a_{\lambda\nu} a_{\mu\rho}\, \delta_{\nu\rho}$$

or

$$\delta^* = a\delta a' \tag{6.3}$$

between the Maxwell coefficients $\delta_{\lambda\mu}$ and $\delta_{\lambda\mu}^*$.

Inverting (6. 3), we obtain

$$c^* = aca' \tag{6.4}$$

It is easy to see that the column matrices of the generalized elastic forces Q and Q^* satisfy the equality

$$Q^* = aQ \tag{6.5}$$

Introducing the column matrices of the generalized elastic coordinates q and q^*, we obtain the elastic motion equations obtained by proceeding from $\mathbf{b}_{i\lambda}$ and $\mathbf{b}_{i\lambda}^*$

$$Mq^{\cdot\cdot} + cq - Q = 0 \tag{6.6}$$

$$Mq^{*\cdot\cdot} + c^* q^* - Q^* = 0 \tag{6.7}$$

This last equation can be transformed into

$$a[M(a'q^*)^{\cdot\cdot} + c(a'q^*) - Q] = 0$$

Comparing this with (6. 6), it is easy to see that the solutions of (6. 6) and (6. 7) are connected by the equality

$$q = a'q^*, \qquad q_\lambda = a_{\mu\lambda}\, q_\mu^*$$

According to (2. 5) the elastic displacements are $\mathbf{V}_i = q_\lambda \mathbf{b}_{i\lambda}$. Simultaneously

$$\mathbf{V}_i = a_{\mu\lambda} q_\mu^* \mathbf{b}_{i\lambda} = q_\mu^* \mathbf{b}_{i\mu}^*$$

Therefore, the elastic displacements $\mathbf{V}_i$ are identical in both cases, and therefore, are independent of the selection of the $\mathbf{b}_{i\lambda}$.

7. Formalized method of determining the unit elastic displacements. Let us consider the transformation of the arbitrary external forces $\mathbf{R}_i$ applied to a rigid system. Let us represent

$$\mathbf{R}_i = \mathbf{S}_i + \mathbf{F}_i \tag{7.1}$$

where the forces $\mathbf{S}_i$ have the same resultant $\mathbf{R}$ and principal moment $\mathbf{L}_c$ relative to the center of mass as the forces $\mathbf{R}_i$

$$\Sigma\mathbf{S}_i = \Sigma\mathbf{R}_i = \mathbf{R}, \qquad \Sigma\mathbf{r}_i\times\mathbf{S}_i = \Sigma\mathbf{r}_i\times\mathbf{R}_i = \mathbf{L}_c \tag{7.2}$$

Let us impose also the requirement that the stress resultants and elastic constraints be zero upon application of the forces $\mathbf{S}_i$. It follows from (7.1) and (7.2) that the forces $\mathbf{F}_i$ producing the stress resultants in the constraints are statically equivalent to zero at each instant.

Let us examine the motion of a system subjected to the force $\mathbf{S}_i$. If there are no stress resultants in the constraints, each point moves without interaction with the rest. Its equation of motion

$$m_i \frac{d^2}{dt^2} \mathbf{r}_i^\circ = \mathbf{S}_i$$

can be represented by using (2.3.1) as

$$m_i\mathbf{r}_i^{\circ\cdot\cdot} + m_i \frac{d}{dt} \boldsymbol{\omega}\times\mathbf{r}_i = \mathbf{S}_i$$

Summing all these equalities with respect to i and taking account of (2.4), we obtain

$$M\mathbf{r}_c^{\circ\cdot\cdot} = \mathbf{R}$$

The equation of motion of the ith point becomes

$$\mathbf{S}_i = \frac{m_i}{M}\mathbf{R} + m_i\boldsymbol{\omega}^\cdot\times\mathbf{r}_i + m_i\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_i)$$

In case the angular velocity of the basis $\mathbf{e}_j'$ is small, the last term in the right side can be neglected and we obtain

$$\mathbf{S}_i = \frac{m_i}{M}\mathbf{R} + m_i\boldsymbol{\omega}^\cdot\times\mathbf{r}_i \tag{7.3}$$

Multiplying both these equalities vectorially on the left by $\mathbf{r}_i$ and summing over all points of the system, by taking account of (7.2) and (2.4) we obtain

$$\Sigma m_i\mathbf{r}_i\times(\boldsymbol{\omega}^\cdot\times\mathbf{r}_i) = \mathbf{L}_c$$

from which $\boldsymbol{\omega}^\cdot$ is easily expressed in terms of $\mathbf{L}_c$, i.e. in terms of the external forces

$$\boldsymbol{\omega}^\cdot = \frac{L_{c1}}{I_1}\mathbf{e}_1' + \frac{L_{c2}}{I_2}\mathbf{e}_2' + \frac{L_{c3}}{I_3}\mathbf{e}_3', \quad \mathbf{L}_c = L_{c1}\mathbf{e}_1' + L_{c2}\mathbf{e}_2 + L_{c3}\mathbf{e}_3'$$

Here I_k are the principal central moments of inertia of the rigid system.

The forces $\mathbf{S}_i$ are determined by the equalities (7.3) and the forces $\mathbf{F}_i$ can be found from (7.1): $\mathbf{F}_i = \mathbf{R}_i - \mathbf{S}_i$.

To construct the formalized process to determine the unit displacements $\mathbf{b}_{i\lambda}$ for an arbitrary system, we determine $3N$ systems of external forces $\mathbf{R}_i^{(\mu)}$ ($i = 1,2,\ldots, N$; $\mu = 1,2,\ldots, 3N$) such that each system consists of the single force $\mathbf{R}_i^{(\mu)}$ applied to any point in the direction of one of the unit vectors $\mathbf{e}_j'$, and there is no identical pair among the systems $\mathbf{R}_i^{(\mu)}$. Evidently all the vectors $\mathbf{f}_{i\lambda}$, $\boldsymbol{\beta}_{i\lambda}$, $\mathbf{b}_{i\lambda}$, $\mathbf{p}_{i\lambda}$ introduced above are linear combinations of the forces $\mathbf{R}_i^{(\mu)}$.

Let us determine the forces $\mathbf{S}_i^{(\mu)}$ for each system of forces $\mathbf{R}_i^{(\mu)}$, and then

$$\mathbf{F}_i^{(\mu)} = \mathbf{f}_{i\mu} \tag{7.4}$$

Furthermore, let us determine the vectors $\boldsymbol{\beta}_{i\lambda}$ by using (3.2), and then by orthonormalizing the $\beta_{i\lambda}$ we find the $\mathbf{b}_{i\lambda}$. Upon execution of these calculations cases are certainly encountered when the scalars $C_{\lambda\lambda}$ in equalities of the type (3.4) turn out to be zero. This means that $\beta_{i\lambda}$ is a linear combination of the $\mathbf{b}_{i\mu}$ $(\lambda > \mu)$. In such cases the vectors $\mathbf{b}_{i\lambda}$ should be omitted and the next vectors $\mathbf{b}_{i,\lambda+1}$ should be considered.

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ON THE STABILITY OF ROTATIONAL MOTION OF A VARIABLE COMPOSITION BODY WITH A GYROSCOPE IN A NEWTONIAN FORCE FIELD

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S. BURALKHIEV
(Chimkent)

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Sufficient conditions are presented for the stability of rotational motion of a variable-mass body in a central Newtonian force field. The equations of body motion around a fixed point are written under assumptions made by M. Sh. Aminov.

The Chetaev method, as well as the V. V. Rumiantsev theorem on the stability of motion relative to part of the variables, are used in investigating the stability of rotational motions of a solid in the Lagrange case.

Let us consider a symmetric body $(A = B)$ of variable mass on whose axis of symmetry a gyroscope with kinetic moment l_0 is placed and there is the center of mass of the body at a distance $Z_c(t)$ from a fixed point O.

If the body is in a central Newtonian force field, the Euler-Poisson equations, under the assumptions considered in [1-3], have the form

$$p^{\cdot} = (1-\delta)\,qr - \nu q + {}^1/_2 a\gamma_2 - \mu(1-\delta)\gamma_2\gamma_3 \qquad (\nu = l_0 / A)$$

$$q^{\cdot} = (\delta - 1)\,pr + \nu p - {}^1/_2 a\gamma_1 + \mu(1-\delta)\gamma_1\gamma_3, \qquad r^{\cdot} = 0 \qquad (\delta = C / A) \qquad (0.1)$$

$$\gamma_1^{\cdot} = r\gamma_2 - q\gamma_3, \qquad \gamma_2^{\cdot} = p\gamma_3 - r\gamma_1, \qquad \gamma_3^{\cdot} = q\gamma_1 - p\gamma_2 \qquad (a = 2MgZ_c / A)$$

Here ν, δ, a are some functions of time, μ is a constant. Evidently, one of the solutions of (0.1)

$$p = q = \gamma_1 = \gamma_2 = 0, \qquad r = r_0, \ \gamma_3 = 1 \qquad (0.2)$$

corresponds to body rotation around an axis of symmetry coinciding with the direction to the center of attraction, at a constant angular velocity.

1. We obtain sufficient conditions for the stability of the motion (0.2) from the equation for the angle of nutation θ. It follows from the equations of motion (0.1)

$$p^2 + q^2 + a\gamma_3 - \mu(1-\delta)\gamma_3^2 - \int_0^t \gamma_3 da - \mu \int_0^t \gamma_3^2 d\delta = C_1 \qquad (1.1)$$